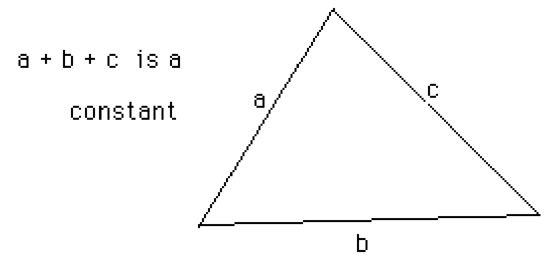
Maximum Area of a Triangle

Problem: Use the Arithmetic Mean -- Geometric Mean Inequality to show that the maximum area of a triangular region with a given perimeter is attained when the triangle is equilateral.



Solution:

Semi-perimeter of the triangle, $S = \frac{a+b+c}{2} \Longrightarrow 2S = a + b + c$

We can find the area using Heron's Formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$

Using AM-GM Inequality,

$$(s-a)(s-b)(s-c) \leq \left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3$$

$$= \left[\frac{3s-(a+b+c)}{3}\right]^3$$

Since, $2S = a + b + c$, we have

$$(s-a)(s-b)(s-c) \leq \left[\frac{3s-2s}{3}\right]^3 = \left[\frac{s}{3}\right]^3 = \frac{s^3}{27}$$

$(s-a)(s-b)(s-c) \leq \frac{s^3}{27}$ with equality when $s - a = s - b = s - c$

Since $a + b + c$ is a constant, then $S = \frac{a+b+c}{2}$ is also a constant

Hence $a = b = c \Longrightarrow$ An Equilateral Triangle.

So, $A = s\left(\frac{s}{3}\right)^3 = \frac{s^4}{27}$.